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Degree Raising for Splines

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1. INTRODUCTION

Polynomial splines play an important role in computer-aided design (cf. [1, 3, 5]) as well as in diverse areas of approximation theory and numerical analysis (cf. [9]). One of the main reasons for their importance is the fact that they can be represented as linear combinations of normalized B-splines (see the following section). This results in convenient stable algorithms for evaluation of the splines as well as their derivatives and integrals (cf. [2, 9]). In addition, there is also a stable and efficient algorithm to convert a B-spline expansion relative to a given knot sequence into an equivalent expansion relative to a refined knot sequence (cf. [1, 3]).

The purpose of this paper is to derive formulae for converting a B-spline

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expansion in terms of B-splines of order m (degree $m-1$) into an equivalent expansion involving B-splines of order $m+1$ (degree m).

There are at least two major applications for the methods presented here. We discuss first an application in the field of *computer-aided design*. Suppose that a curve is represented by a B-spline series of order m with a given set of knots, and that a designer desires added flexibility in manipulating the shape of the curve. One way of achieving this is to add additional knots (cf. [1, 3]). With the tools presented here, we now have an alternative way of adding flexibility—by raising the degree of the spline. For more details and explicit algorithms, see [4].

A second application for our degree raising algorithms is to the problem of combining two splines whose B-spline expansions are in terms of B-splines of differing orders. This problem arises, for example, if we are attempting to find all points where two given splines curves $s(x)$ and $\tilde{s}(x)$ cross each other. This is equivalent to finding the zeros of the difference $\Delta = s - \tilde{s}$, and to work with this expression numerically, we need to write it as a single B-spline expansion.

The paper is organized as follows. In Section 2 we introduce some notation and state the main result. Several preliminary results as well as the proof of the main theorem are contained in Section 3. In Section 4 we discuss certain discrete B-splines which play a role. Section 5 contains examples, and finally, Section 6 is devoted to remarks.

2. The Degree Raising Formula

Given a positive integer m, points $\tau_1 < \cdots < \tau_k$ and integers $1 \leqslant m_i \leqslant m$, $i = 1, \ldots, k$, suppose that

$$y_1 \leqslant \cdots \leqslant y_{n+m} = \overbrace{\tau_1, \ldots, \tau_1}^{m_1}, \ldots, \overbrace{\tau_k, \ldots, \tau_k}^{m_k}, \tag{2.1}$$

where $n+m = \sum_{i=1}^{k} m_i$. Suppose s is an mth-order spline written in the form

$$s(x) = \sum_{i=1}^{n} c_i N_i^m(x), \tag{2.2}$$

where N_i^m are the usual normalized B-splines of order m defined by

$$N_i^m(x) = (-1)^m (y_{i+m} - y_i)[y_i, \ldots, y_{i+m}](x-y)_+^{m-1}. \tag{2.3}$$

Our aim in this paper is to write the spline s in (2.2) as a B-spline expansion of order $m+1$. In order to do this we must first introduce a new knot sequence where each knot τ_i has multiplicity m_i+1 instead of m_i, $i=1,\dots,k$. The rationale for this is clear if we recall that a spline s of order m has $m-m_i-1$ continuous derivatives in a neighborhood of the knot τ_i, and thus to make the spline $\tilde{s}$ of order $m+1$ agree identically with s, we must give $\tilde{s}$ an (m_i+1)-tuple knot at τ_i. Suppose $\cdots \leqslant \tilde{y}_0 \leqslant \tilde{y}_1 \leqslant \cdots$ is such that

$$\tilde{y}_1 \leqslant \cdots \leqslant \tilde{y}_{\tilde{n}+m+1} = \overbrace{\tau_1,\dots,\tau_1}^{m_1+1},\dots,\overbrace{\tau_k,\dots,\tau_k}^{m_k+1}, \tag{2.4}$$

where $\tilde{n}=n+k-1$. For each i, let $\tilde{N}_i^{m+1}$ be the normalized B-spline associated with the extended knot sequence $\tilde{y}_i \leqslant \cdots \leqslant \tilde{y}_{i+m+1}$.

We can now state the main result of the paper.

THEOREM 2.1. *Let $s=\sum_{i=1}^n c_i N_i^m$ be a B-spline expansion with respect to B-splines $\{N_i^m\}_1^n$ associated with a knot sequence $\{y_i\}_1^{n+m}$ as in* (2.1). *Then s can be written as a B-spline series with respect to B-splines $\{\tilde{N}_i^{m+1}\}_1^{\tilde{n}}$ associated with the knot sequence $\{\tilde{y}_i\}_1^{\tilde{n}+m+1}$ as in* (2.4). *In particular,*

$$s=\sum_{j=1}^{\tilde{n}} \tilde{c}_j \tilde{N}_j^{m+1}, \tag{2.5}$$

where

$$\tilde{c}_j=\frac{1}{m}\sum_{i=1}^{n} c_i \Lambda_i^m(j), \qquad j=1,\dots,\tilde{n}, \tag{2.6}$$

and

$$\Lambda_i^m(j)=(y_{i+m}-y_i)[y_i,\dots,y_{i+m}]\, D\Psi_j^{m+1}, \tag{2.7}$$

$$\Psi_j^{m+1}(y)=(y-\eta_j)_+^0 \prod_{v=1}^{m} (y-\tilde{y}_{j+v}). \tag{2.8}$$

Here η_j are arbitrary points satisfying $\tilde{y}_j<\eta_j<\tilde{y}_{j+m+1}$, for $j=1,\dots,\tilde{n}$. *The sum in* (2.6) *involves at most m terms. Indeed, let $l_j=\min\{l\geqslant 1\colon y_{l+m}\geqslant \tilde{y}_{j+m+1}\}$ and $r_j=\max\{r\leqslant n\colon y_r\leqslant \tilde{y}_j\}$. Then*

$$\tilde{c}_j=\frac{1}{m}\sum_{i=l_j}^{r_j} c_i \Lambda_i^m(j). \tag{2.9}$$

The proof of this theorem will be given in the following section.

3. Proof of Theorem 2.1

We begin this section with a lemma concerning one-sided splines.

Lemma 3.1. *Given* $1 \leqslant i \leqslant k$, *suppose that* $0 \leqslant r \leqslant m_i$ *and that* $\tilde{y}_j < \eta_j < \tilde{y}_{j+m+1}$, *all* j. *Then for any* x,

$$D_y^r (y-x)_+^m \big|_{y=\tau_i} = \sum_{j=l-m}^{l} D^r \phi_j^{m+1}(\tau_i)(\tau_i - \eta_j)_+^0 \, \tilde{N}_j^{m+1}(x), \tag{3.1}$$

where l *is such that* $\tilde{y}_l \leqslant x < \tilde{y}_{l+1}$, *and where*

$$\phi_j^{m+1}(y) = \prod_{\nu=1}^{m} (y - \tilde{y}_{j+\nu}), \qquad j = 1, ..., \tilde{n}. \tag{3.2}$$

Proof. Let p be such that $\tau_i = \tilde{y}_p = \cdots = \tilde{y}_{p+m_i}$. Applying D_y^r to (3.2), we obtain

$$D^r \phi_j^{m+1}(y) = \sum \sum_{j+1 \leqslant i_1 < \cdots < i_{m-r} \leqslant j+m} \cdots \sum (y - \tilde{y}_{i_1}) \cdots (y - \tilde{y}_{i_{m-r}}).$$

Now since $\tau_i = \tilde{y}_p = \cdots = \tilde{y}_{p+m_i}$ and $0 \leqslant r \leqslant m_i$, each term in this sum contains the factor $(y - \tau_i)$ provided that j is such that $p + m_i - m \leqslant j \leqslant p - 1$. It follows that

$$D^r \phi_j^{m+1}(\tau_i) = 0, \qquad r = 0, 1, ..., m_i, \; j = p + m_i - m, ..., p - 1. \tag{3.3}$$

To complete the proof we consider two cases.

Case 1 $(x \geqslant \tau_i)$. Clearly the left-hand side of (3.1) is zero in this case. Moreover, by the support properties of the *B*-splines, the right-hand side of (3.1) is given by

$$\left[\sum_{j=l-m}^{p-1} + \sum_{j=p}^{l} \right] [D^r \phi_j^{m+1}(\tau_i)(\tau_i - \eta_j)_+^0 \, \tilde{N}_j^{m+1}(x)].$$

Using (3.3) we see that each term in the first sum is zero since $x \geqslant \tau_i$ implies $l - m \geqslant p + m_i - m$. On the other hand, each term in the second sum is also zero since $(\tau_i - \eta_j)_+^0 = 0$, $j = p, ..., l$.

Case 2 $(x < \tau_i)$. In this case we have $l \leqslant p - 1$, and thus using the well-known Marsden identity

$$(y-x)^m = \sum_{j=l-m}^{l} \phi_j^{m+1}(y) \, \tilde{N}_j^{m+1}(x), \tag{3.4}$$

we get

$$D_y^r(y-x)_+^m\,|_{y=\tau_i}=D_y^r(y-x)^m\,|_{y=\tau_i}=\sum_{j=l-m}^{l} D^r\phi_j^{m+1}(\tau_i)\,\tilde{N}_j^{m+1}(x)$$

$$=\sum_{j=l-m}^{l} D^r\phi_j^{m+1}(\tau_i)(\tau_i-\eta_j)_+^0\,\tilde{N}_j^{m+1}(x),$$

where the last equation holds because of (3.3) and the fact that

$$(\tau_i-\eta_j)_+^0=1 \qquad \text{for} \quad j=l-m,\dots,p+m_i-m-1. \quad \blacksquare$$

We can now give a formula for the expansion of a single B-spline N_i^m in terms of the $\tilde{N}_j^{m+1}$'s.

THEOREM 3.2. *Given $1\leqslant i\leqslant n$, let p_i and q_i be such that $\tau_{p_i}=y_i$ and $\tau_{q_i}=y_{i+m}$. Then*

$$N_i^m=\frac{1}{m}\sum_{j=i+p_i-1}^{i+q_i-1} \Lambda_i^m(j)\,\tilde{N}_j^{m+1}, \tag{3.5}$$

where the Λ_i^m's are defined in (2.7)–(2.8).

Proof. Let $\mu_{p_i},\dots,\mu_{q_i}$ with $\mu_{p_i}+\cdots+\mu_{q_i}=m+1$ be such that

$$y_i\leqslant\cdots\leqslant y_{i+m}=\overbrace{\tau_{p_i},\dots,\tau_{p_i}}^{\mu_{p_i}}<\cdots<\overbrace{\tau_{q_i},\dots,\tau_{q_i}}^{\mu_{q_i}}. \tag{3.6}$$

Then (cf. [9]), there exist coefficients β_{vr} (independent of f) with

$$[y_i,\dots,y_{i+m}]f=\sum_{v=p_i}^{q_i}\sum_{r=1}^{\mu_v}\beta_{vr}D^{r-1}f(\tau_v),$$

for all sufficiently smooth f. Now using (3.1) we obtain (cf. [3])

$$\begin{aligned} mN_i^m(x)/(y_{i+m}-y_i)&=[y_i,\dots,y_{i+m}]\,D_y(y-x)_+^m\\ &=\sum_{v=p_i}^{q_i}\sum_{r=1}^{\mu_v}\beta_{vr}D_y^r(y-x)_+^m\,|_{y=\tau_v}\\ &=\sum_{v=p_i}^{q_i}\sum_{r=1}^{\mu_v}\beta_{vr}\sum_{j=-\infty}^{\infty}D^r\phi_j^{m+1}(\tau_v)(\tau_v-\eta_j)_+^0\,\tilde{N}_j^{m+1}(x)\\ &=\sum_{j=-\infty}^{\infty}[y_i,\dots,y_{i+m}]\,D_y\Psi_j^{m+1}(y)\,\tilde{N}_j^{m+1}(x)\\ &=\frac{\sum_{j=-\infty}^{\infty}\Lambda_i^m(j)\,\tilde{N}_j^{m+1}(x)}{(y_{i+m}-y_i)}. \end{aligned}$$

To complete the proof, we now examine when $\Lambda_i^m(j) \neq 0$. First we note that by the linear independence of the B-splines, the coefficient $\Lambda_i^m(j)$ in the above expansion of N_i^m in terms of the $\tilde{N}_j^{m+1}$'s can be nonzero only if the support of $\tilde{N}_j^{m+1}$ is contained in the support of N_i^m. Thus, we have

$$\Lambda_i^m(j) \neq 0 \text{ implies } y_i \leqslant \tilde{y}_j < \tilde{y}_{j+m+1} \leqslant y_{i+m}. \tag{3.7}$$

We can get a more precise statement by observing that $\Lambda_i^m(j) \neq 0$ implies $\tilde{N}_j^{m+1}$ has at least as many continuous derivatives as N_i^m at y_i and y_{i+m}. Now the spline N_i^m has $m-1-\mu_{p_i}$ continuous derivatives at y_i and $m-1-\mu_{q_i}$ continuous derivatives at y_{i+m}. Now by the definition (2.4) of the $\tilde{y}$'s

$$\tau_{p_i} = \tilde{y}_{i+\mu_{p_i}-m_{p_i}+p_i-1} = \cdots = \tilde{y}_{i+\mu_{p_i}+p_i-1} < \tilde{y}_{i+\mu_{p_i}+p_i}$$

$$\tau_{q_i} = \tilde{y}_{i+m-\mu_{q_i}+q_i} = \cdots = \tilde{y}_{i+m-\mu_{q_i}+m_{q_i}+q_i} > \tilde{y}_{i+m-\mu_{q_i}+q_i-1}.$$

It follows that $\tilde{N}_j^{m+1}$ exhibits the same continuities as or higher continuities than N_i^m at y_i and y_{i+m} only if $i+p_i-1 \leqslant j \leqslant i+q_i-1$. ∎

We can now establish our main result.

Proof of Theorem 2.1. Substituting (3.5) into (2.2), we get

$$s = \frac{1}{m} \sum_{i=1}^{n} c_i \sum_{j=1}^{\tilde{n}} \Lambda_i^m(j)\, \tilde{N}_j^{m+1} = \sum_{j=1}^{\tilde{n}} \tilde{c}_j \tilde{N}_j^{m+1}$$

with $\tilde{c}_j$ given by (2.6). Finally, by (3.7), $\Lambda_i^m(j) = 0$ unless $l_j \leqslant i \leqslant r_j$. ∎

4. Discrete Splines

In order to turn Theorem 2.1 into a useful algorithm, we need to study the coefficients $\Lambda_i^m(j)$ appearing in (2.6) in detail. We begin by recalling some results on *discrete B-splines*. Suppose $\cdots \leqslant y_1 \leqslant y_2 \leqslant \cdots$ and suppose that $\cdots \leqslant \tilde{y}_1 \leqslant \tilde{y}_2 \leqslant \cdots$ is a refinement obtained by repeating some of the y's. Then associated with these knot sequences, the discrete B-splines are defined by

$$\begin{aligned} \alpha_i^m(j) &= (y_{i+m} - y_i)[y_i,\ldots, y_{i+m}]\, \Psi_j^m(y), && \text{if } y_i < y_{i+m} \\ &= 0, && \text{otherwise,} \end{aligned} \tag{4.1}$$

where

$$\Psi_j^m(y) = \prod_{\nu=1}^{m-1} (y - \tilde{y}_{j+\nu})(y - \eta_j)_+^0.$$

Here η_j can be an arbitrary point satisfying $\tilde{y}_j \leqq \eta_j < \tilde{y}_{j+m}$ provided that $\tilde{y}_j < \tilde{y}_{j+m}$. If $\tilde{y}_j = \tilde{y}_{j+m}$, then we take $\eta_j = \tilde{y}_j$, and in this case $\alpha_i^m(j) = N_i^m(\tilde{y}_j)$. For some results on discrete splines, see [3, 6] and references therein. In particular, it was shown in [3, p. 97] that the α's can be computed recursively starting with

$$\begin{aligned} \alpha_i^1(j) &= 1, \qquad \text{if } \ y_i \leqslant \tilde{y}_j < y_{i+1} \\ &= 0, \qquad \text{otherwise,} \end{aligned} \tag{4.2}$$

and using the formula

$$\alpha_i^m(j) = \frac{(\tilde{y}_{j+m-1} - y_i)}{(y_{i+m-1} - y_i)}\, \alpha_i^{m-1}(j) + \frac{(y_{i+m} - \tilde{y}_{j+m-1})}{(y_{i+m} - y_{i+1})}\, \alpha_{i+1}^{m-1}(j). \tag{4.3}$$

It was also shown in [3] that

$$\alpha_i^m(j) \geqslant 0 \tag{4.4}$$

$$\alpha_i^m(j) = 0, \qquad \text{if } \tilde{y}_j < y_i \text{ or if } y_{i+m} < \tilde{y}_{j+m-1} \tag{4.5}$$

$$\sum_{i=1}^{n} \alpha_i^m(j) = 1, \qquad \text{for all } j \text{ such that } y_m \leqslant \tilde{y}_j < y_{n+1} \tag{4.6}$$

$$N_i^m(x) = \sum_j \alpha_i^m(j)\, \tilde{N}_j^m(x). \tag{4.7}$$

Precise conditions under which the α's are positive were obtained in [6]. For later use, we now determine directly the set of positive α's in the special case where the knots are chosen as in Section 2.

LEMMA 4.1. *Let* $\{y_i\}_1^{n+m}$ *and* $\{\tilde{y}_i\}_1^{\tilde{n}+m+1}$ *be as in* (2.1)–(2.4). *Given* $1 \leqslant i \leqslant n$, *let* p_i *and* q_i *be as in Theorem* 3.2. *Then*

$$\alpha_i^m(j) > 0 \qquad \textit{for} \quad j = i + p_i, \ldots, i + q_i - 1. \tag{4.8}$$

Proof. If $y_i = y_{i+m}$ there is nothing to prove. Suppose now that $y_i < y_{i+m}$. We proceed by induction on m. The assertion is obvious for $m = 1$. Now since $y_i < \tilde{y}_{j+m-1} \leqslant y_{i+m}$ for $j = i + p_i, \ldots, i + q_i - 1$, it follows that in (4.3) the factors are all nonnegative. But then by the inductive hypothesis,

$$\alpha_i^m(j) \geqslant \frac{(\tilde{y}_{j+m-1} - y_i)}{(y_{i+m-1} - y_j)}\, \alpha_i^{m-1}(j) > 0, \qquad j = i + p_i, \ldots, i + q_i - 2.$$

The same statement holds for $j = i + q_i - 1$ if $y_{i+m-1} = y_{i+m}$. Now suppose that $y_{i+m-1} < y_{i+m}$. Then since $\tilde{y}_{i+q_i+m-2} = y_{i+m-1}$, using the inductive hypothesis again, we obtain

$$\alpha_i^m(i+q_i-1) \geqslant \frac{(y_{i+m} - \tilde{y}_{i+q_i+m-2})}{(y_{i+m} - y_{i+1})} \alpha_{i+1}^{m-1}(i+q_i-1) > 0.$$

This completes the proof. ∎

In the remainder of this section, we establish similar properties for the $\Lambda_i^m(j)$'s. First we extend the definition of the Λ's to more general knot sequences. Suppose $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_{n+m}$ is as in (2.1), but with arbitrary m_i's. Now suppose that $\tilde{y}_1 \leqslant \cdots \leqslant \tilde{y}_{\tilde{n}+m+1}$ is the refinement defined in (2.4). Then we define

$$\begin{aligned} \Lambda_i^m(j) &= 0, & &\text{if} \quad y_i = y_{i+m} \\ &= \text{as in (2.7)}, & &\text{otherwise}, \end{aligned} \tag{4.9}$$

provided that $\tilde{y}_j < \tilde{y}_{j+m+1}$. If $\tilde{y}_j = \tilde{y}_{j+m+1}$, then we may use the definition (2.7), but we must take $\eta_j = \tilde{y}_j$. In this case $\Lambda_i^m(j) = mN_i^m(\tilde{y}_j)$. This equality also holds when $\tilde{y}_j = \tilde{y}_{j+m} < \tilde{y}_{j+m+1}$.

We now establish a useful recursion relation for computing the Λ's.

THEOREM 4.2. *For all i and j,*

$$\begin{aligned} \Lambda_i^m(j) = &\frac{(\tilde{y}_{j+m} - y_i)}{(y_{i+m-1} - y_i)} \Lambda_i^{m-1}(j) \\ &+ \frac{(y_{i+m} - \tilde{y}_{j+m})}{(y_{i+m} - y_{i+1})} \Lambda_{i+1}^{m-1}(j) + \alpha_i^m(j). \end{aligned} \tag{4.10}$$

Thus, the Λ's can be computed recursively starting with

$$\Lambda_i^1(j) = \alpha_i^1(j). \tag{4.11}$$

Proof. If $y_i = y_{i+m}$, there is nothing to prove. Suppose now that $y_i < y_{i+m}$. If $\tilde{y}_j = \cdots = \tilde{y}_{j+m}$, then since $\Lambda_i^m(j) = mN_i^m(\tilde{y}_j)$, all of the assertions follow from properties of the usual B-splines (cf. [9]). Suppose now that $\tilde{y}_j < \tilde{y}_{j+m}$. Then by definition (2.7),

$$\begin{aligned} \Lambda_i^m(j) &= (y_{i+m} - y_i)[y_i, \ldots, y_{i+m}] \, D_y \Psi_j^{m+1}(y) \\ &= (y_{i+m} - y_i)[y_i, \ldots, y_{i+m}] \sum_{\mu=1}^{m} \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{m} (y - \tilde{y}_{j+\nu})(y - \eta_j)_+^0 \end{aligned}$$

$$= \sum_{\mu=1}^{m-1} (y_{i+m} - y_i)[y_i,\dots, y_{i+m}](y - y_{j+m})$$
$$\times \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{m-1} (y - \tilde{y}_{j+\nu})(y - \eta_j)_+^0 + \alpha_i^m(j). \tag{4.12}$$

Now writing

$$\Phi_j^m(y) = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{m-1} (y - \tilde{y}_{j+\nu})(y - \eta_j)_+^0,$$

and applying Leibniz' rule for divided differences (cf. [9, p. 50]),

$$\begin{aligned}
&[y_i,\dots, y_{i+m}](y - \tilde{y}_{j+m})\, \Phi_j^m(y) \\
&\quad = ([y_{i+1},\dots, y_{i+m}] + (y_i - \tilde{y}_{j+m})[y_i,\dots, y_{i+m}])\, \Phi_j^m(y) \\
&\quad = \left(\frac{(y_{i+m} - \tilde{y}_{j+m})}{(y_{i+m} - y_i)} [y_{i+1},\dots, y_{i+m}] - \right. \\
&\qquad \left. \frac{(y_i - \tilde{y}_{i+m})}{(y_{i+m} - y_i)} [y_i,\dots, y_{i+m-1}]\right) \Phi_j^m(y).
\end{aligned}$$

Substituting in (4.12) leads to (4.10). Equation (4.11) follows directly fron the definition (4.9). ∎

The recursion relation (4.10) can be used to establish additiona interesting properties of Λ's. The following theorem provides analogs of th properties (4.8) and (4.6) of discrete B-splines.

THEOREM 4.3. *Suppose that* $\{y_i\}_1^{n+m}$ *and* $\{\tilde{y}_i\}_1^{\tilde{n}+m+1}$ *are knot sequence as in* (2.1) *and* (2.4), *respectively. Then for all* $1 \leqslant i \leqslant n$,

$$\Lambda_i^m(j) > 0, \qquad \text{for} \quad j = i + p_i - 1,\dots, i + q_i - 1 \tag{4.13}$$
$$\Lambda_i^m(j) = 0, \qquad \textit{otherwise}, \tag{4.14}$$

where p_i *and* q_i *are such that* $\tau_{p_i} = y_i$ *and* $\tau_{q_i} = y_{i+m}$. *Moreover,*

$$\sum_{i=1}^{n} \Lambda_i^m(j) = m, \qquad \textit{for all } 1 \leqslant j \leqslant \tilde{n}. \tag{4.15}$$

We have already established (4.14) in the proof of Theorem 3.2. Fo $i + p_i - 2 \leqslant j \leqslant i + q_i - 1$, the positivity assertion (4.13) follows from th recursion (4.10) coupled with the fact (cf. (4.11)) that $\Lambda_i^1(j) \geqslant 0$ while b

(4.8) we have $\alpha_i^m(j) > 0$ for the stated j's. For $j = i + p_i - 1$, (4.13) follows from the fact that

$$D^{m-\mu_i} N_i^m(y_i) = \frac{1}{m} A_i^m(i + p_i - 1)\, D^{m-\mu_i} \tilde{N}_{i+p_i-1}^{m+1}(y_i) > 0.$$

To prove (4.15), we sum both sides of (3.5) for $i = 1$ to $i = n$ to obtain

$$m = \sum_{j=1}^{\tilde{n}} \left(\sum_{i=1}^{n} A_i^m(j) \right) \tilde{N}_j^{m+1} = m \sum_{j=1}^{\tilde{n}} \tilde{N}_j^{m+1},$$

and the result follows by the linear independence of the $\tilde{N}_j^{m+1}$'s. ∎

5. Examples

In this section we present two examples to illustrate Theorem 2.1.

EXAMPLE 5.1. Suppose $m = 2$ and $y_i = (i-1)/2$, $i = 1, 2, 3$. Then for this knot sequence, there is a single linear B-spline N_1^2. Let $s = N_1^2$.

Discussion. In this case we have $n = 1$ and $c_1 = 1$. In order to write s in terms of quadratic B-splines, we take the knots $\tilde{y}_1 = \tilde{y}_2 = 0$, $\tilde{y}_3 = \tilde{y}_4 = \frac{1}{2}$, and $\tilde{y}_5 = \tilde{y}_6 = 1$. Then Theorem 2.1 gives $\tilde{c}_1 = \tilde{c}_3 = \frac{1}{2}$ and $\tilde{c}_2 = 1$; i.e., $N_1^2 = \frac{1}{2}\tilde{N}_1^3 + \tilde{N}_2^3 + \frac{1}{2}\tilde{N}_3^3$. (See Fig. 1.) ∎

EXAMPLE 5.2. Suppose $m = 4$ and that $y_i = (i-1)/4$, $i = 1, \ldots, 5$. Then with this knot sequence, there is a single cubic B-spline N_1^4. Suppose that $s = N_1^4$, i.e., $c_1 = 1$.

Discussion. In order to write s as a combination of quartic B-splines, we take the knots $\{\tilde{y}_1, \ldots, \tilde{y}_{10}\}$ to be $\{0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1, 1\}$. Then $\tilde{n} = 5$

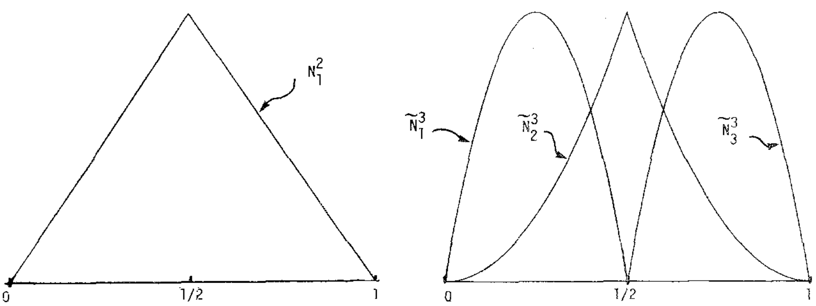

FIG. 1. The splines of Example 5.1.

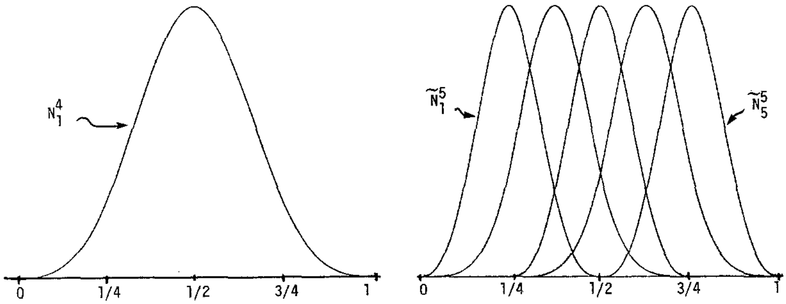

FIG. 2. The splines of Example 5.2.

and the coefficients of s in terms of the quintic B-splines associated with the $\tilde{y}$'s are $(\frac{1}{12}, \frac{1}{2}, \frac{5}{6}, \frac{1}{2}, \frac{1}{12})$. Figure 2 shows s and the quartic B-splines which must be combined to express it. ∎

6. REMARKS

1. The problem of raising the degree of a polynomial in Taylor form is trivial. The formula for raising the degree of a polynomial written in terms of the Bernstein basis has been mentioned in a variety of papers. For example, in connection with computer-aided design, see [5].

2. In some applications it is useful to use periodic B-splines rather than the usual B-splines discussed here (for example, in modelling closed curves with the Bezier method). The methods presented here can be modified to apply to periodic splines, or alternatively can be applied directly without modification since expansions in terms of periodic B-splines can be written in terms of the usual B-splines (with an appropriate periodic extension of the knots—cf. [9]).

3. The A_i^m's appearing in Theorem 2.1 are clearly related to the discrete splines α_i^m defined in (4.1). Indeed (cf. the proof of Theorem 4.2), they are sums of certain discrete splines.

4. In Theorem 3.2 we showed how to write an mth-order B-spline defined on a set of knots as in (2.1) in terms of $(m+1)$st-order B-splines defined on a set of knots as in (2.4). It turns out that it is also possible to wrtite the mth-order B-spline in terms of $(m+1)$st-order B-splines which are *not* defined on consecutive knots drawn from (2.4). For example, using an obvious notation, the formula on page 227 of [8] becomes $3 * N(x \mid y_0, y_1, y_2) = N(x \mid y_0, y_0, y_1, y_2) + N(x \mid y_0, y_1, y_1, y_2) + N(x \mid y_0, y_1, y_2, y_2)$. This is an expansion of a linear B-spline in terms of three quadratic B-splines. The usefulness of these kinds of degree-raising formulae is under further study.

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